

A COUPLE OF REAL HYPERBOLIC DISC BUNDLES OVER SURFACES

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ABSTRACT. Applying the techniques developed in [AGG], we construct new real hyperbolic manifolds whose underlying topology is that of a disc bundle over a closed orientable surface. By the Gromov-Lawson-Thurston conjecture [GLT], such bundles $M \rightarrow S$ should satisfy the inequality $|eM/\chi S| \leq 1$, where eM stands for the Euler number of the bundle and χS , for the Euler characteristic of the surface. In this paper, we construct new examples that provide a maximal value of $|eM/\chi S| = \frac{3}{5}$ among all known examples. The former maximum, belonging to Feng Luo [Luo], was $|eM/\chi S| = \frac{1}{2}$.

Dedicated to Krolík

1. Introduction

Topologically or differentially, every open disc bundle $M \rightarrow S$ over a closed connected orientable surface S can be completely characterized by two numbers: the Euler characteristic χS of the surface and the Euler number eM of the bundle, i.e., the number of self-intersections of a section of the bundle. Note that, taking an unramified finite cover of S and pullbacking the bundle, one gets the same value of $|eM/\chi S|$.

The conjecture of Gromov, Lawson, and Thurston [GLT, p. 28] suggests a numerical criterion deciding whether a bundle can be equipped with a complete real hyperbolic geometry.

1.1. GLT-conjecture. A disc bundle $M \rightarrow S$ over a closed connected orientable surface S of genus $g \geq 2$ admits a complete real hyperbolic structure iff $|eM/\chi S| \leq 1$.

In [AGG], Conjecture 1.1 was extended (with the same bound) to the complex hyperbolic case.

1.2. Known results. The best proven upper bound belongs to Misha Kapovich [Kap] who showed that $|eM| \leq \exp\left(\exp(10^8|\chi S|)\right)$ for any complete real hyperbolic 4-manifold homotopically equivalent to a closed orientable surface; so, without actually using the fact that it admits a disc bundle structure. (In this case, eM stands for the self-intersection of the generator of H_2M represented by a homotopy equivalence $S \rightarrow M$.) It is worthwhile mentioning that, in such settings, Nicholaas H. Kuiper [Kui2] constructed examples with $|eM/\chi S| > \frac{2}{\sqrt{3}} > 1$.

In the other direction, the best results belong to N. H. Kuiper [Kui1] and to Feng Luo [Luo]. In [Kui1, Theorem 6, p. 68], it was constructed a series of disc bundles admitting complete real hyperbolic geometry with any rational value of $|eM/\chi S|$ in the interval $[0, \frac{1}{3}]$. (Though, we are not sure that this result is literally correct as there are a few miscalculations in the exposition.) F. Luo constructed an example with a maximal known (before our paper) value $|eM/\chi S| = \frac{1}{2}$. Since the surface S in F. Luo's example has genus 2, taking a finite unramified cover of S , one gets examples satisfying the relation $|eM/\chi S| = \frac{1}{2}$ with S of an arbitrary genus $g \geq 2$.

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1.3. Main result. Using the ideas of [AGG] and a coordinate free approach to hyperbolic geometry exposed in [AGr2, AGr3], we construct 3 new examples of disc bundles with $|eM/\chi S| = \frac{3}{5}$.

Two of them have $eM = 12$ with S of genus 11. The third one has $eM = 24$ with S of genus 21. The first two come from right-angled necklace polyhedra with 84 codimension 1 faces and 21 cycles of codimension 2 faces. The third one has a similar flavor with 164 codimension 1 faces and 41 cycles of codimension 2 faces.

In fact, we construct 3 orbifolds whose 4-sheeted covers provide the above manifolds. Each orbifold comes from a right-angled necklace polyhedron P symmetric with respect to a regular elliptic isometry r of order 24 for the first 2 examples and of order 44 for the third one. The face-pairing of these polyhedra are reflections in their totally geodesic codimension 1 faces.

For the other examples of disc bundles of a similar type, see Theorem 3.10.

2. Construction

We start the construction by fixing a regular elliptic isometry r of indicated order n . Denote by P_1 and P_2 its totally geodesic r -stable planes. The planes intersect orthogonally at the unique r -fixed point b .

Next, we pick a generic totally geodesic hyperplane $H_0 \subset \overline{\mathbb{H}}_{\mathbb{R}}^4$. Geometrically, the pair r, H_0 is given by the distances from b to the intersections $H_0 \cap P_1$ and $H_0 \cap P_2$. So, it can be described by means of 2 convenient real parameters x_1, x_2 responsible for these distances.

Then we copy the hyperplane, $H_i := r^i H_0$. The conditions that $C_i := H_i \cap H_{i+1} \not\subset \partial \overline{\mathbb{H}}_{\mathbb{R}}^4$ and that the other pairs of hyperplanes H_i and H_j , $i - j \not\equiv_n \pm 1$, are ultraparallel, i.e., $H_i \cap H_j = \emptyset$, is equivalent by Lemma 3.1 to a finite number of inequalities linear in x_1, x_2 . Thus, we arrive at a convex region $R \subset \mathbb{R}^2(x_1, x_2)$. In what follows, we assume $(x_1, x_2) \in R$.

Intersecting those closed half-spaces limited by the H_i 's that contain the point b , we get a convex polyhedron P bounded by closed solid cylinders $B_i \subset H_i$ and by a piece $\partial_1 P \subset \partial \overline{\mathbb{H}}_{\mathbb{R}}^4$ of the absolute bounded by a torus $T \subset \partial \overline{\mathbb{H}}_{\mathbb{R}}^4$. In turn, the solid cylinder B_i is bounded inside H_i by the ultraparallel totally geodesic planes C_{i-1}, C_i and by a cylinder inside T ; such cylinders form the torus T . Denoting $\partial_0 P := \bigcup_{i=1}^n B_i$, we see that $\partial_0 P$ is a solid torus bounded by T and that $\partial P = \partial_0 P \sqcup_T \partial_1 P$.

Every solid cylinder B_i is fibred by totally geodesic planes called *slices* of B_i . Indeed, the geodesic segment Γ_i that joins the closest points in C_{i-1} and in C_i lists the fibres in question: through any $p \in \Gamma_i$, we have, inside H_i , a totally geodesic plane orthogonal to Γ_i . Note that C_{i-1} and C_i are among the slices; they are the *initial* and the *final* slices. Denote by M_i the *middle slice*, i.e., the one passing through the middle point of Γ_i .

Every solid cylinder B_i is fibred as well by closed geodesic segments called *strings* of B_i . They can be described as follows. Take any totally geodesic plane F such that $\Gamma_i \subset F \subset H_i$. The intersection $F \cap B_i$ is bounded in F by the geodesics $F \cap C_{i-1}$ and $F \cap C_i$, both orthogonal to Γ_i , and by two arcs on the absolute. A *string* of B_i is the segment of a line, inside some F , equidistant from the geodesic containing the segment Γ_i . The segment Γ_i and the mentioned two arcs on the absolute are among the strings of B_i . Clearly, the reflection σ_i in the middle slice M_i of B_i stabilizes any string of B_i and interchanges the endpoints of the string.

Pick a point $q_0 \in \partial \overline{\mathbb{H}}_{\mathbb{R}}^4 \cap C_0$ and let $q_0 \in s_n \subset B_n$ be the string of B_n that contains q_0 . Then $q_{n-1} := \sigma_n q_0 \in \partial \overline{\mathbb{H}}_{\mathbb{R}}^4 \cap C_{n-1}$ is the other endpoint of s_n . Next, we take the string s_{n-1} of B_{n-1} such that $q_{n-1} \in s_{n-1} \subset B_{n-1}$, and so on. Finally, we get a simple curve $s := s_1 \cup s_2 \cup \dots \cup s_n$ with the endpoints $q_0 \in \partial \overline{\mathbb{H}}_{\mathbb{R}}^4 \cap C_0$ and $q'_0 := \sigma_1 \sigma_2 \dots \sigma_n q_0 \in \partial \overline{\mathbb{H}}_{\mathbb{R}}^4 \cap C_0$, where $\partial \overline{\mathbb{H}}_{\mathbb{R}}^4 \cap B_i \supset s_i$ is a string of B_i for all $1 \leq i \leq n$. We call such a curve $s \subset T$ the *string* of P generated by $q_0 \in \partial \overline{\mathbb{H}}_{\mathbb{R}}^4 \cap C_0$.

2.1. Lemma (cf. [AGG, Lemma 2.25, p. 4317]). *If H_0 is orthogonal to P_1 (in terms of the parameters, this means that $x_2 = 0$), then any string of P is closed and contractible in $\partial_1 P$. So, $\partial_1 P$ is a solid torus and the slice bundle of $\partial_0 P$ is extendable to P in this case.*

Proof. When H_0 is orthogonal to P_1 , the intersection $Q := P \cap P_1$ is a regular n -gon centred at b in the hyperbolic plane P_1 . The polyhedron P is simply the union of all those totally geodesic planes orthogonal to P_1 that pass through a point of Q . The slices of $\partial_0 P$ are built over the points of the boundary ∂Q . Thus, we get the slice bundle of $\partial_0 P$ extended to P .

The isometry σ_i is the trivial extension of the reflection in the middle point of the corresponding side of Q . Hence, the isometry $\sigma_1 \dots \sigma_n$ is a trivial extension of an elliptic isometry of P_1 with the fixed point $C_0 \cap P_1$. In other words, the restriction $\sigma_1 \dots \sigma_n|_{C_0}$ is the identity, implying that any string of P is closed.

In order to visualize a contraction of a closed string $s \subset T$ in $\partial_1 P$, one can simply shrink the n -gon Q (say, keeping its r -rotational symmetry about b).

The fact that $\partial_1 P$ is a solid torus follows from the Dehn lemma ■

If any string of the polyhedron P is closed, we say that P is *fibred*. As we saw, this is equivalent to $\sigma_1 \dots \sigma_n|_{C_0} = 1_{C_0}$. Denote $\sigma := \sigma_0$. Then $\sigma_i = r^i \sigma r^{-1}$ and $\sigma_1 \dots \sigma_n = (r\sigma)^n$ because $r^n = 1$. Since C_0 is $r\sigma$ -stable, we conclude that P is fibred iff $r\sigma|_{C_0}$ is an elliptic isometry of C_0 whose order divides n .

It follows from the connectedness of the region R and from Lemmas 2.1 and 3.1 that $\partial_1 P$ is a solid torus. By [AGG, Lemma 2.19, p. 4312], P is topologically a closed 4-ball and the slice bundle of $\partial_0 P$ is extendable to P .

Suppose that P is fibred. Take a couple of disjoint strings $s, s' \subset T$ of P . Since s and s' are contractible inside P with 2-discs $D, D' \subset P$, $s = \partial D$ and $s' = \partial D'$, one can calculate the algebraic number of intersections of D and D' . This number eP (clearly independent of the choice of s, s') is the *Euler number* of the fibred polyhedron P .

Suppose that a fibred polyhedron P is equipped with face-pairing for the codimension 1 faces B_i and that the conditions of Poincaré's polyhedron theorem (PPT) are satisfied (see, for example, [AGr1]). Then we obtain a disc bundle M over a 2-dimensional orbifold S and eP is the Euler number of this bundle because the (face-pairing) isometries preserve the slice bundles and the string bundles of the B_i 's. (See, for instance, [BoS] for a treatment of bundles over orbifolds. Another option is to glue a few copies of P forming a fundamental polyhedron for a manifold and to note that eP and χS get multiplied by the number of copies.)

A closed curve $c \subset T$ that generates the group $H_1(\partial_0 P, \mathbb{Z})$ is said to be *trivializing* if $[c] = 0$ in $H_1(\partial_1 P, \mathbb{Z})$. In the case considered in Lemma 2.1, any string of P is trivializing.

2.2. Lemma [AGG, Remark 2.22, p. 4314]. *Let P be a fibred polyhedron, let $T \supset s$ be a string of P , and let $T \supset c$ be a trivializing curve of P . Then $eP = \#s \cap c$. In other words, $[s] = eP \cdot [g]$ in the group $H_1(\partial_1 P, \mathbb{Z})$, where $g := \partial \mathbb{H}_{\mathbb{R}}^4 \cap C_0$ ■*

Now we get a tool to measure the Euler number eM . Given fibred polyhedron satisfying the conditions of PPT, one can deform it into a 'plane' one (dealt with in Lemma 2.1) because the region R is convex. At the beginning of the deformation, the Euler number eP of a 'plane' polyhedron equals 0 by Lemmas 2.1 and 2.2. During the deformation, we keep track of how many times the polyhedron becomes fibred, i.e., how many times a string of the polyhedron becomes closed. Of course, counting these events, we should take care of the signs. So, it is better to say that the Euler number of the fibred polyhedron at the end of the deformation equals the algebraic number of times it was fibred during the deformation, including the last moment and not including the initial 'plane' moment.

We apply this method at the end of the proof of Theorem 3.9. The count is simple there because the chosen deformation provides a monotonic evolution of a string.

3. Calculation

Let b_1, b_2, b_3, b_4, b be an orthonormal basis of signature $- - - +$ in an \mathbb{R} -linear space V equipped with a symmetric bilinear form $\langle -, - \rangle$. Then the real hyperbolic space $\mathbb{H}_{\mathbb{R}}^4$, its absolute $\partial\mathbb{H}_{\mathbb{R}}^4$, and $\overline{\mathbb{H}}_{\mathbb{R}}^4 := \mathbb{H}_{\mathbb{R}}^4 \sqcup \partial\mathbb{H}_{\mathbb{R}}^4$ are known to be identified respectively with

$$\mathbb{H}_{\mathbb{R}}^4 := \{p \in \mathbb{P}_{\mathbb{R}}V \mid \langle p, p \rangle > 0\}, \quad \partial\mathbb{H}_{\mathbb{R}}^4 := \{p \in \mathbb{P}_{\mathbb{R}}V \mid \langle p, p \rangle = 0\}, \quad \overline{\mathbb{H}}_{\mathbb{R}}^4 := \{p \in \mathbb{P}_{\mathbb{R}}V \mid \langle p, p \rangle \geq 0\}.$$

Pick some numbers $k, m, n \in \mathbb{N}$ such that n is even and $1 < k < m < \frac{n}{2}$. Denote by

$$r := \begin{bmatrix} c_1 & -s_1 & 0 & 0 & 0 \\ s_1 & c_1 & 0 & 0 & 0 \\ 0 & 0 & c_m & -s_m & 0 \\ 0 & 0 & s_m & c_m & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad p_0 := \begin{bmatrix} \sqrt{x_1} \\ 0 \\ \sqrt{x_2} \\ 0 \\ \sqrt{x_1+x_2-1} \end{bmatrix}, \quad 0 < x_1, \quad 0 \leq x_2, \quad 1 < x_1 + x_2,$$

the regular elliptic isometry of $\mathbb{H}_{\mathbb{R}}^4$ with the unique fixed point $b \in \mathbb{H}_{\mathbb{R}}^4$ and a point $p_0 \in \mathbb{P}_{\mathbb{R}}V \setminus \overline{\mathbb{H}}_{\mathbb{R}}^4$, both written in the above basis, where x_1, x_2 are some real parameters subject to the displayed inequalities, $c_i := \cos \frac{2i\pi}{n}$, and $s_i := \sin \frac{2i\pi}{n}$. Clearly, $r^n = 1$, $r \in \text{SO } V$, and $\langle p_0, p_0 \rangle = -1$. For any $i \in \mathbb{Z}$, denote

$$p_i := r^i p_0 = \begin{bmatrix} c_i & -s_i & 0 & 0 & 0 \\ s_i & c_i & 0 & 0 & 0 \\ 0 & 0 & c_{mi} & -s_{mi} & 0 \\ 0 & 0 & s_{mi} & c_{mi} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{x_1} \\ 0 \\ \sqrt{x_2} \\ 0 \\ \sqrt{x_1+x_2-1} \end{bmatrix} = \begin{bmatrix} c_i \sqrt{x_1} \\ s_i \sqrt{x_1} \\ c_{mi} \sqrt{x_2} \\ s_{mi} \sqrt{x_2} \\ \sqrt{x_1+x_2-1} \end{bmatrix}.$$

Obviously,

$$g_i := \langle p_0, p_i \rangle = (1 - c_i)x_1 + (1 - c_{mi})x_2 - 1.$$

Denote by $H_i \subset \overline{\mathbb{H}}_{\mathbb{R}}^4$ the totally geodesic hyperplane corresponding to the \mathbb{R} -linear subspace $p_i^\perp \leq V$. Let $C_i := H_i \cap H_{i+1}$.

3.1. Lemma. *The conditions that $C_i \not\subset \partial\overline{\mathbb{H}}_{\mathbb{R}}^4$ and that $H_i \cap H_j = \emptyset$ for all i, j such that $i - j \not\equiv_n \pm 1$ are equivalent to the inequalities*

$$(3.2) \quad 0 < x_1, \quad 0 \leq x_2, \quad 1 < x_1 + x_2, \quad (1 - c_1)x_1 + (1 - c_m)x_2 < 2 < (1 - c_i)x_1 + (1 - c_{mi})x_2,$$

for $2 \leq i \leq \frac{n}{2}$. The convex region $R \subset \mathbb{R}^2(x_1, x_2)$ given by the inequalities (3.2) is nonempty because it contains the segment $S := (\frac{2}{1-c_2}, \frac{2}{1-c_1}) \times 0 \subset R$.

Proof. The condition that $C_i \not\subset \partial\overline{\mathbb{H}}_{\mathbb{R}}^4$ is known to be equivalent to $-1 < \langle p_i, p_{i+1} \rangle < 1$ and the condition that $H_i \cap H_j = \emptyset$ is known to be equivalent to $\langle p_i, p_j \rangle \notin [-1, 1]$, taking into account that $\langle p_k, p_k \rangle = -1$ for all k . Since $0 \leq g_i + 1$ and $0 < g_1 + 1$, due to the symmetry related to the action of r , we arrive at the inequalities (3.2). For $x_2 = 0$, the inequalities take the form $1 < x_1$ and $(1 - c_1)x_1 < 2 < (1 - c_2)x_1$, i.e., the form $1 < \frac{2}{1-c_2} < x_1 < \frac{2}{1-c_1}$ with $-1 < c_2 < c_1$ ■

In the sequel, we assume $(x_1, x_2) \in R$.

Obviously, $C_i \cap C_j = \emptyset$ unless $i \equiv_n j$. Therefore, the ultraparallel planes $C_{i-1}, C_i \subset H_i \simeq \overline{\mathbb{H}}_{\mathbb{R}}^3$ limit in H_i a solid cylinder B_i , and these cylinders form a solid torus $\partial_0 P := \bigcup_{i=1}^n B_i$ bounded by a torus

$T \subset \partial\overline{\mathbb{H}}_{\mathbb{R}}^4$ as claimed in Section 2.

All the C_i 's are in a same closed half-space of $\overline{\mathbb{H}}_{\mathbb{R}}^4$ limited by a hyperplane H_j as, otherwise, there would exist some C_{i-1} and C_i , both disjoint with H_j , in different half-spaces, which would cause an impossible intersection $H_i \cap H_j \neq \emptyset$.

Let P stand for the intersection of those closed half-spaces limited by the H_i 's that contain the point b ; the B_i 's are codimension 1 faces of P and the C_i 's are codimension 2 faces of P . By Lemma 2.1, $\partial_1 P$ is a solid torus, P is a closed 4-ball, and $\partial P = \partial_0 P \sqcup_T \partial_1 P$.

Denote by σ the reflexion in the middle slice M_0 of B_0 and by τ , the reflection in H_0 . Clearly, $\sigma_i := r^i \sigma r^{-i}$ is the reflection in the middle slice M_i of B_i and $\tau_i := r^i \tau r^{-i}$ is the reflection in H_i .

3.3. Lemma. *Suppose that $g_1 = 0$, i.e.,*

$$(3.4) \quad (1 - c_1)x_1 + (1 - c_m)x_2 = 1.$$

Then $(\tau_{i+1}\tau_i)^2 = 1$ for all i and the polyhedron $\mathbb{H}_{\mathbb{R}}^4 \cap P$ endowed with the face-pairing isometries τ_i (identifying every codimension 1 face B_i with itself) satisfies the conditions of Poincaré's polyhedron theorem.

Proof. The hyperplanes H_i and H_{i+1} are orthogonal along C_i because $\langle p_i, p_{i+1} \rangle = g_1 = 0$. As the reflection τ_i is given by the rule $\tau_i : v \mapsto v + 2\langle v, p_i \rangle p_i$, the equality $(\tau_{i+1}\tau_i)^2 = 1$ follows straightforwardly from $\langle p_i, p_{i+1} \rangle = 0$ and $\langle p_i, p_i \rangle = \langle p_{i+1}, p_{i+1} \rangle = -1$.

Since τ_i sends the interior of P into the exterior of P and every geometric cycle of codimension 2 faces of P has length 4 and, therefore, total angle 2π , the conditions of PPT are satisfied (see [AGr1, Theorem 3.2, p. 303] and [AGr1, Proposition 2.1, p. 300]) ■

3.5. Lemma. *Let $U := \mathbb{R}p_0 + \mathbb{R}p_1$ and $W := \mathbb{R}p_0 + \mathbb{R}(p_1 - p_{n-1})$. Then C_0 and M_0 correspond respectively to the \mathbb{R} -linear subspaces U^\perp and W^\perp . The isometries σ and $r\sigma$ are given by the rules*

$$(3.6) \quad v \mapsto v + 2\langle v, p_0 \rangle p_0 + \frac{\langle v, p_1 - p_{n-1} \rangle}{g_2 + 1} (p_1 - p_{n-1}),$$

$$(3.7) \quad r\sigma v = rv + 2\langle v, p_0 \rangle p_1 + \frac{\langle v, p_1 - p_{n-1} \rangle}{g_2 + 1} (p_2 - p_0).$$

The \mathbb{R} -linear subspace U is $r\sigma$ -stable and $\begin{bmatrix} 0 & 1 \\ -1 & 2g_1 \end{bmatrix}$ is the matrix of $r\sigma|_U$ in the basis p_0, p_1 . The point $f_0 := (1 - g_1)b + \langle b, p_0 \rangle (p_0 + p_1) \in \mathbb{H}_{\mathbb{R}}^4 \cap C_0$ is a fixed point of $r\sigma$.

Proof. By definition, C_0 corresponds to U^\perp . In other words, C_0 corresponds to $p_0^\perp \cap (p_1 + g_1 p_0)^\perp$ and, similarly, C_{n-1} corresponds to $p_0^\perp \cap (p_{n-1} + g_1 p_0)^\perp$, where $p_1 + g_1 p_0, p_{n-1} + g_1 p_0 \in p_0^\perp$. Since

$$\langle p_1 + g_1 p_0, p_1 + g_1 p_0 \rangle = \langle p_1 + g_1 p_0, p_1 \rangle = -1 + g_1^2 < 0,$$

$$\langle p_{n-1} + g_1 p_0, p_{n-1} + g_1 p_0 \rangle = \langle p_{n-1} + g_1 p_0, p_{n-1} \rangle = -1 + g_1^2 < 0,$$

$$\langle p_1 + g_1 p_0, p_{n-1} + g_1 p_0 \rangle = \langle p_1 + g_1 p_0, p_{n-1} \rangle = g_2 + g_1^2 > 0,$$

the middle point of Γ_0 equals $m_0 := (p_1 + g_1 p_0) + (p_{n-1} + g_1 p_0) = p_1 + p_{n-1} + 2g_1 p_0$ and M_0 corresponds to $p_0^\perp \cap ((p_1 + g_1 p_0) - (p_{n-1} + g_1 p_0))^\perp$.

Clearly, p_0 and $p_1 - p_{n-1}$ are orthogonal. Hence, the rule (3.6) acting as $v \mapsto v$ for any $v \in W^\perp$ and as $v \mapsto -v$ for any $v \in W$ in view of $\langle p_1 - p_{n-1}, p_1 - p_{n-1} \rangle = -2g_2 - 2$ defines the isometry σ . The formula (3.7) is now immediate. It implies the equalities $r\sigma p_0 = -p_1$ and $r\sigma p_1 = p_0 + 2g_1 p_1$ providing the indicated matrix.

Taking $rb = b$ into account, we see that $\langle b, p_i \rangle$ is independent of i , hence, $\langle f_0, p_0 \rangle = \langle f_0, p_1 \rangle = 0$. Therefore, $g_1 < 1$ implies $\langle f_0, f_0 \rangle = (1 - g_1)\langle f_0, b \rangle = (1 - g_1)^2 + 2(1 - g_1)\langle b, p_0 \rangle^2 > 0$, i.e., $f_0 \in \mathbb{H}_{\mathbb{R}}^4 \cap C_0$. Finally,

$$r\sigma f_0 = rf_0 - \frac{\langle f_0, p_{n-1} \rangle}{g_2 + 1} (p_2 - p_0) = (1 - g_1)b + \langle b, p_0 \rangle (p_1 + p_2) -$$

$$-\frac{(1-g_1)\langle b, p_0 \rangle + \langle b, p_0 \rangle (g_1 + g_2)}{g_2 + 1}(p_2 - p_0) = (1-g_1)b + \langle b, p_0 \rangle (p_0 + p_1) = f_0 \blacksquare$$

3.8. Lemma. *The isometry $r\sigma|_{C_0}$ of C_0 is a rotation by a about f_0 , where*

$$\cos a = \frac{(1-c_1^2)c_mx_1 + c_1(1-c_m^2)x_2}{(1-c_1^2)x_1 + (1-c_m^2)x_2}.$$

Proof. The isometry $r\sigma$ preserves orientation. By Lemma 3.5, the isometry $r\sigma|_U$ preserves orientation. Consequently, the isometry $r\sigma|_{C_0}$ also preserves orientation. By Lemma 3.5, it has to be a rotation by some angle a about f_0 . Since $\text{tr}(r\sigma|_U) = 2g_1$ by Lemma 3.5 and

$$\text{tr}(r\sigma) = \text{tr } r + 2g_1 + \frac{\langle p_2 - p_0, p_1 - p_{-1} \rangle}{g_2 + 1} = 1 + 2c_1 + 2c_m + 2g_1 + \frac{g_1 - g_3}{g_2 + 1}$$

by (3.7), we obtain

$$\cos a = c_1 + c_m + \frac{g_1 - g_3}{2(g_2 + 1)} = c_1 + c_m + \frac{(c_3 - c_1)x_1 + (c_{3m} - c_m)x_2}{2(1 - c_2)x_1 + 2(1 - c_{2m})x_2}.$$

Taking $c_2 = 2c_1^2 - 1$, $c_3 = 4c_1^3 - 3c_1$, $c_{2m} = 2c_m^2 - 1$, and $c_{3m} = 4c_m^3 - 3c_m$ into account, we get

$$\cos a = c_1 + c_m + \frac{(c_1^3 - c_1)x_1 + (c_m^3 - c_m)x_2}{(1 - c_1^2)x_1 + (1 - c_m^2)x_2} = \frac{(1 - c_1^2)c_mx_1 + c_1(1 - c_m^2)x_2}{(1 - c_1^2)x_1 + (1 - c_m^2)x_2} \blacksquare$$

3.9. Theorem. *Suppose that the solution of the system*

$$\begin{cases} (1 - c_1)x_1 + (1 - c_m)x_2 = 1 \\ (1 - c_1^2)(c_k - c_m)x_1 = (1 - c_m^2)(c_1 - c_k)x_2 \end{cases}$$

satisfies the inequalities $2 < (1 - c_i)x_1 + (1 - c_{mi})x_2$ for all $2 \leq i \leq \frac{n}{2}$, where $k, m, n \in \mathbb{N}$, n is even, $1 < k < m < \frac{n}{2}$, and $c_i := \cos \frac{2i\pi}{n}$. Then there exists a disc bundle $M \rightarrow S$ over a closed connected orientable surface S admitting a complete real hyperbolic geometry such that $|eM/\chi S| = \frac{4m-4k}{n-4}$.

Proof. First, we observe that the solution clearly satisfies the inequalities $0 < x_1$, $0 < x_2$, and $(1 - c_1)x_1 + (1 - c_m)x_2 < 2$. The inequality $1 < x_1 + x_2$ follows from the inequality $2 < (1 - c_i)x_1 + (1 - c_{mi})x_2$ with $i = \frac{n}{2}$ because $c_i = -1$ and $-c_{mi} < 1$. In other words, we get a point $(x_1, x_2) \in R$ in the region R .

Let $p(t) := (x_1(t), x_2(t))$, $t \in [0, 1]$, be a linearly parameterized path in R that joins a point in the segment $S \subset R$ (see Lemma 3.1) with the point $p(1) = (x_1, x_2)$. The function $a(t)$ is continuous and, by Lemma 3.8, the function $\cos a(t)$ has a form $\cos a(t) = \frac{a_1 t + a_2}{a_3 t + a_4}$ for some $a_1, a_2, a_3, a_4 \in \mathbb{R}$. By Lemma 3.8, $\cos a(0) = \frac{(1-c_1^2)c_mx_1(0)}{(1-c_1^2)x_1(0)} = c_m$ and $\cos a(1) = \frac{(1-c_1^2)c_mx_1 + c_1(1-c_m^2)x_2}{(1-c_1^2)x_1 + (1-c_m^2)x_2} = c_k$ due to the second equation of the system. Hence, the function $\cos a(t)$ is not constant and is therefore monotonic. Consequently, the function $a(t)$ is monotonic.

It was understood in Section 2 that the polyhedron $P(t)$ is fibred iff $r\sigma|_{C_0}$ is a periodic isometry whose order divides n , i.e., iff $\cos a(t) = c_j$ for some $j \in \mathbb{Z}$. Since $a(t)$ is monotonic, $\cos a(0) = c_m$, and $\cos a(1) = c_k$, we conclude that $eP(1) = m - k$.

By Lemma 3.3, the polyhedron $P(1)$ satisfies the conditions of PPT due to the first equation of the system. It remains to observe that the Euler characteristic of the corresponding orbifold S equals $\chi S = \frac{n}{4} - \frac{n}{2} + 1 = -\frac{n-4}{4} \blacksquare$

3.10. Calculation. Taking $(k, m, n) := (2, 5, 24)$ (or $(k, m, n) := (3, 6, 24)$ or $(k, m, n) := (5, 11, 44)$), one can check the 11 (or 11 or 21) inequalities of Theorem 3.9 thus arriving at $|eM/\chi S| = \frac{3}{5}$ ■

References

- [AGG] S. Anan'in, C. H. Grossi, N. Gusevskii, *Complex hyperbolic structures on disc bundles over surfaces*, Int. Math. Res. Not. **2011** (2011), no. 19, 4295–4375, <http://arxiv.org/abs/math/0511741>
- [AGr1] S. Anan'in, C. H. Grossi, *Yet another Poincaré's polyhedron theorem*, Proc. Edinburgh Math. Soc., **54** (2011), 297–308, <http://arxiv.org/abs/math/0812.4161>
- [AGr2] S. Anan'in, C. H. Grossi, *Coordinate-free classic geometries*, Moscow Math. J. **11** (2011), no. 4, 633–655, <http://arxiv.org/abs/math/0702714>
- [AGr3] S. Anan'in, C. H. Grossi, *Differential geometry of grassmannians and the Plücker map*, Central European J. of Math. **10** (2012), no. 3, 873–884, <http://arxiv.org/abs/0907.4470>
- [BoS] F. Bonahom, L. Siebenmann, *The classification of Seifert fibered 3-orbifolds*, in *Low Dimensional Topology*, edited by R. Fenn, LMS Lecture Notes in Science **95**, New York: Cambridge University Press, 1985, 258 pp.
- [GLT] M. Gromov, H. B. Lawson Jr., W. Thurston, *Hyperbolic 4-manifolds and conformally flat 3-manifolds*, Inst. Hautes Études Sci. Publ. Math., no. 68 (1988), 27–45
- [Kap] M. Kapovich, *On hyperbolic 4-manifolds fibered over surfaces*, preprint (1993) <http://www.math.ucdavis.edu/~kapovich/eprints.html>
- [Kui1] N. H. Kuiper, *Hyperbolic 4-manifolds and tessellations*, Inst. Hautes Études Sci. Publ. Math., no. 68 (1988), 47–76
- [Kui2] N. H. Kuiper, *Fairly symmetric hyperbolic manifolds*, in *Geometry and Topology of Submanifolds, II* (1990), World Sci. Publ., 165–204
- [Luo] F. Luo, *Constructing conformally flat structures on some Seifert fibred 3-manifolds*, Math. Ann. **294** (1992), no. 3, 449–458

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